Chapter 9: Radiating Systems, Multipole Fields and Radiation

An Overview of Chapters on EM Waves : (covered in this course)

	source term in wave equation	boundary
Ch. 7	none	plane wave in ∞ space or in two semi-∞ spaces separated by the x-y plane
Ch. 8	none	conducting walls
Ch. 9	$\mathbf{J}, \rho \sim e^{-i\omega t}$ prescribed, as in an antenna	outgoing wave to ∞
Ch. 10	$\mathbf{J}, \rho \sim e^{-i\omega t}$ induced by incident EM waves, as in the case of scattering of a plane wave by a dielectric object.	outgoing wave to ∞
Ch. 14	moving charges, such as electrons in a synchrotron	outgoing wave to ∞

9.6 Spherical Wave Solutions of the Scalar Wave Equation

Spherical Bessel Functions and Hankel functions : Although this chapter deals with radiating systems, here we first solve the scalar source-free wave equation in the spherical coordinate syatem. The purpose is to obtain a complete set of spherical Bessel funtions and Hankel functions, with which we will expand the fields produced by the sources.

The scalar source-free wave equation is [see (6.32)]

$$\nabla^2\psi(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t)=0 \tag{9.77}$$

Let $\psi(\mathbf{x},t)=\int_{-\infty}^{\infty}\psi(\mathbf{x},\omega)e^{-i\omega t}d\omega$ (9.78)

$\Rightarrow$ Each Fourier component satisfies the Helmholtz wave eq.

$$(\nabla^2+k^2)\psi(\mathbf{x},\omega)=0, \tag{9.79}$$

where $k\equiv\frac{\omega}{c}$

9.6 Spherical Wave Solutions… (*continued*)

In spherical coordinates, $(\nabla^2 + k^2)\Phi = 0$ is written

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} + k^2\Phi = 0$$

Let $\Phi = U(r)P(\theta)Q(\phi)$, we obtain

$$PQ\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) + UQ\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + UP\frac{1}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2} + k^2UPQ = 0$$

Multiply by $\dfrac{r^2\sin^2\theta}{UPQ}$

The only term with ϕ-dependence, so this term must be a constant. Let it be $-m^2$.

$$\sin^2\theta[\underbrace{\frac{1}{U}\frac{d}{dr}(r^2\frac{dU}{dr}) + k^2r^2}_{=l(l+1)} + \frac{1}{P\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})] + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{=-m^2} = 0$$

Dividing all terms by $\sin^2\theta$, we see that this is the only term with r-dependence. So it must be a constant. Let it be $l(l+1)$.

$$\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

Thus, as in Sec. 3.1 of lecture notes,

$$P = P_\ell^m(\cos\theta),\ Q_\ell^m(\cos\theta);\ Q = e^{im\phi}, e^{-im\phi} \Rightarrow\ PQ = Y_{lm}(\theta,\phi)$$

rejected because of divergence at $\theta = \pm\pi$

9.6 Spherical Wave Solutions… (*continued*)

$U(r)$ is governed by $\frac{d}{dr}(r^2\frac{dU}{dr})+k^2r^2U=l(l+1)U$. Rewrite U as $f_l(r)$. Then, $\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}\right]f_l(r)=0$ (9.81)

Let $f_l(r)=\frac{1}{r^{1/2}}u_l(r)\Rightarrow\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(l+1/2)^2}{r^2}\right]u_l(r)=0$ (9.83)

$\Rightarrow\ u_l(r)=J_{l+\frac{1}{2}}(kr),\ N_{l+\frac{1}{2}}(kr)$ [Bessel functions of fractional order]

$\Rightarrow\ f_l(r)=\frac{1}{r^{1/2}}J_{l+\frac{1}{2}}(kr),\ \ \frac{1}{r^{1/2}}N_{l+\frac{1}{2}}(kr)$

Define $\begin{cases} j_l(kr)=\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr) \\ n_l(kr)=\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}N_{l+\frac{1}{2}}(kr) \end{cases}$ and $\begin{cases} h_l^{(1)}(kr)=j_l(kr)+in_l(kr) \\ h_l^{(2)}(kr)=j_l(kr)-in_l(kr) \end{cases}$

spherical Bessel functions

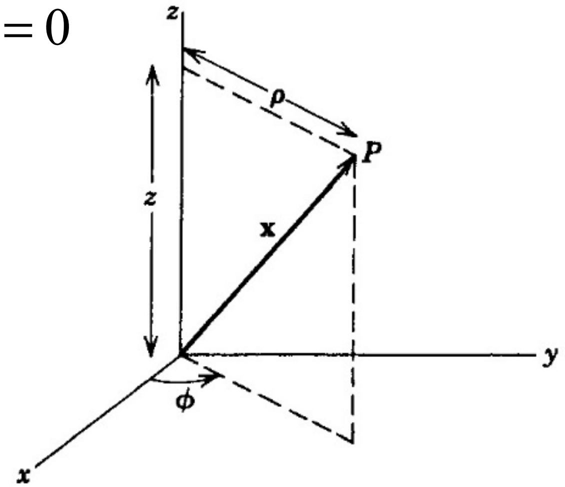
spherical Hankel functions

$$\Rightarrow\psi(\mathbf{x},\omega)=\sum_{lm}\left[A_{lm}^{(1)}h_l^{(1)}(kr)+A_{lm}^{(2)}h_l^{(2)}(kr)\right]Y_{lm}(\theta,\phi)\quad [k=\tfrac{\omega}{c}]\quad (9.92)$$

Review

3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions

$$\nabla^2\Phi(\mathbf{x}) = 0 \Rightarrow \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$



Let $\phi(\mathrm{x}) = R(\rho)Q(\varphi)Z(z)$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 Z}{\partial z^2} - k^2 Z = 0 \Rightarrow Z = e^{\pm kz} \\ \dfrac{\partial^2 Q}{\partial\phi^2} + \nu^2 Q = 0 \Rightarrow Q = e^{\pm i\nu\phi} \\ \dfrac{\partial^2 R}{\partial\rho^2} + \dfrac{1}{\rho}\dfrac{\partial R}{\partial\rho} + \left(k^2 - \dfrac{\nu^2}{\rho^2}\right)R = 0 \Rightarrow R = J_\nu(k\rho),\ N_\nu(k\rho) \end{cases}$$

where J_ν and N_ν are Bessel functions of the first and second kind, respectively (see following pages).

$$\Rightarrow \Phi = \begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix} \begin{Bmatrix} e^{i\nu\phi} \\ e^{-i\nu\phi} \end{Bmatrix} \begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix} \qquad (3)$$

Review

3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions (*continued*)

Bessel Functions : If we let $x = k\rho$, the equation for R takes the standard form of the Bessel equation,

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\nu^2}{x^2}\right)R=0 \qquad (3.77)$$

with solutions $J_\nu(x)$ and $N_\nu(x)$, from which we define the Hankel functions:

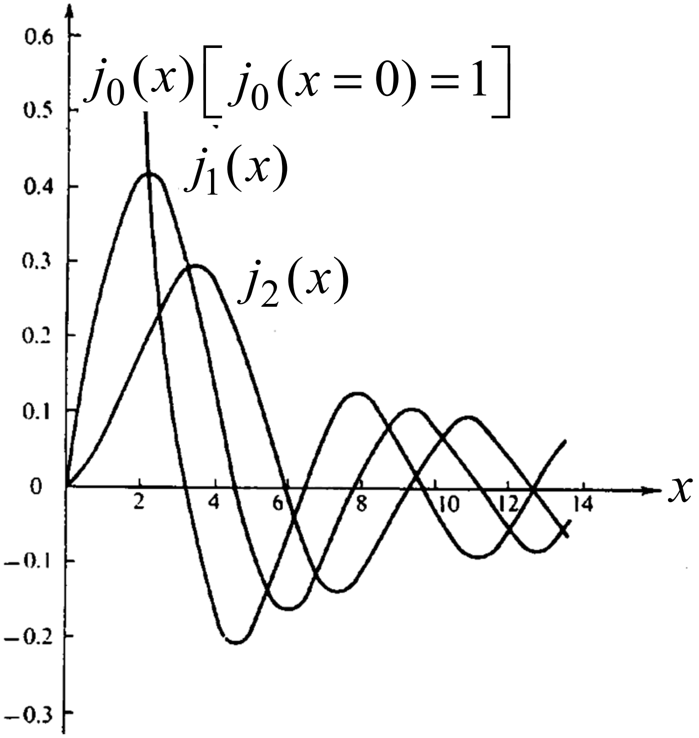
$$\begin{cases} H_\nu^{(1)}(x)=J_\nu(x)+iN_\nu(x) \\ H_\nu^{(2)}(x)=J_\nu(x)-iN_\nu(x) \end{cases} \qquad (3.86)$$

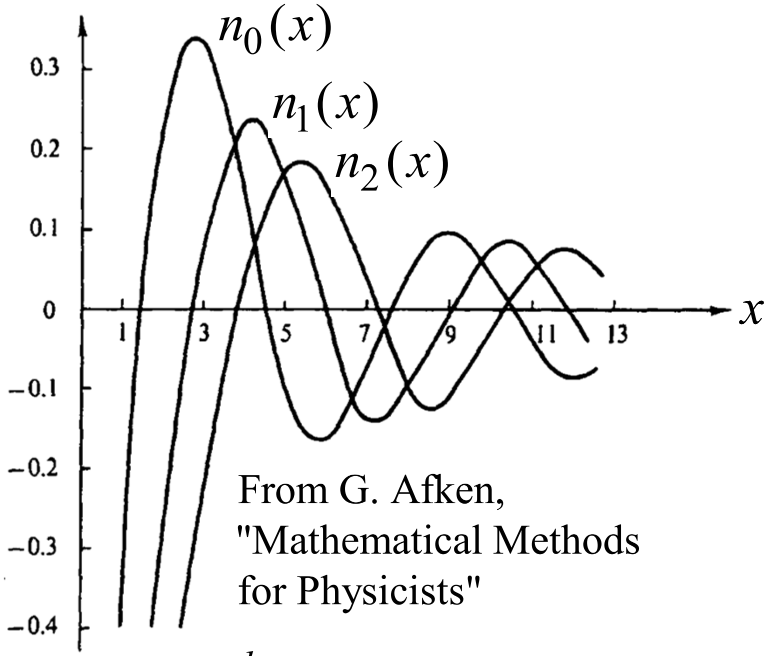
and the modified Bessel functions (Bessel functions of imaginary argument)

$$\begin{cases} I_\nu(x)=i^{-\nu}J_\nu(ix) & (3.100) \\ K_\nu(x)=\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ix) & (3.101) \end{cases}$$

See Jackson pp. 112-116, Gradshteyn & Ryzhik, and Abramowitz & Stegun for properties of these special functions.

9.6 Spherical Wave Solutions… (*continued*)





From G. Afken, "Mathematical Methods for Physicists"

$$j_l(x)\xrightarrow{x\gg l}\frac{1}{x}\sin\left(x-\frac{l\pi}{2}\right)$$

$$j_l(x)\xrightarrow{x\ll 1,\ l}\frac{x^l}{(2l+1)!!}\left[1-\frac{x^2}{2(2l+3)}+\cdots\right]$$

$$n_l(x)\xrightarrow{x\gg l}-\frac{1}{x}\cos\left(x-\frac{l\pi}{2}\right)$$

$$n_l(x)\xrightarrow{x\ll 1,\ l}-\frac{(2l-1)!!}{x^{l+1}}\left[1-\frac{x^2}{2(1-2l)}+\cdots\right]$$

$h_l^{(1)}(x)\xrightarrow{x\gg l}(-i)^{l+1}\frac{e^{ix}}{x}$ [$\Rightarrow$ spatial dependence of spherical waves.]

See Jackson pp. 426-427 for further properties of j_l, n_l, $h_l^{(1)}$, and $h_l^{(2)}$.

Expansion of the Green function : Solution of the Green equation

$$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{6.36}$$

is given by (derived in Sec. 6.4.)

$$G(\mathbf{x},\mathbf{x}')=\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\quad\left[\begin{array}{l}\text{in infinite space and for outgoing-}\\ \text{wave boundary condition.}\end{array}\right] \tag{6.40}$$

We may solve (6.36) in the same way as in Sec. 3.9, i.e., write

$$G(\mathbf{x},\mathbf{x}')=\sum_{lm} g_l(r,r')Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi),$$

solve for $g_l(r,r')$ for $r>r'$ and $r<r'$ [where $\delta(\mathbf{x}-\mathbf{x}')=0$], and then apply boundary conditions at $r=0,\ r=\infty,$ and $r=r'$. The result is

$$G(\mathbf{x},\mathbf{x}')=4\pi ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Equating the two expressions above for $G(\mathbf{x},\mathbf{x}')$, we obtain

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}=4\pi ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi), \tag{9.98}$$

where $r_<$ and $r_>$ are, respectively, the smaller and larger of r and r'.

9.6 Spherical Wave Solutions… (*continued*)

Summary of Differential Equations and Solutions:

Source-free D.E.	Laplace eq. $\nabla^2\boldsymbol{\Phi} = 0$	Helmholtz eq. $\left(\nabla^2 + k^2\right)\psi = 0$	Wave Eq. $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0$
Solutions Cartesian cylindrical spherical	$e^{i\alpha x},\ e^{i\beta y},\ e^{\sqrt{\alpha^2+\beta^2}\,z}$ (Sec. 2.9) $J_m(kr),\ e^{im\theta},\ e^{kz}$ (Sec. 3.7) $Y_{lm}(\theta,\phi),\ r^l$ (Secs. 3.1, 3.2)	$e^{ik_x x},\ e^{ik_y y},\ e^{ik_z z}$, etc. (Sec. 8.4) $J_m\left(\sqrt{\frac{\omega^2}{c^2} - k_z^2}\right), e^{im\theta}, e^{ik_z z}$ (Sec. 8.7) $Y_{lm}(\theta,\phi),\ j_l(kr),\ n_l(kr)$ (Sec. 9.6)	$\begin{Bmatrix}\mathbf{A}(\mathbf{x},t)\\ \boldsymbol{\Phi}(\mathbf{x},t)\end{Bmatrix} =$ $\iint d^3x'dt'\,\frac{\delta\left[t' - \left(t - \frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\begin{Bmatrix}\mu_0\mathbf{J}(\mathbf{x}',t')\\ \frac{\rho(\mathbf{x}',t')}{\varepsilon_0}\end{Bmatrix}$
D.E. with a point source	$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: $G(\infty) = 0$	$(\nabla^2 + k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: outgoing wave	$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G^+(\mathbf{x},t,\mathbf{x}',t')$ $= -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ b.c.: outgoing wave
Solutions (Green functions)	$G = \frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$G = \frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.40)]	$G^+(\mathbf{x},t,\mathbf{x}',t') = \frac{\delta\left[t' - \left(t - \frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.44)]
Series expansion of Green function	Eqs. (3.70), (3.148), (3.168)	Eq. (9.98)	

9.1 Radiation of a Localized Oscillating Source

Review of Inhomogeneous Wave Equations and Solutions :

$$\begin{cases} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\rho/\varepsilon_0 & (6.15) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J} & (6.16) \end{cases}$$

[in free space, Φ and $\mathbf{A}$ satisfy Lorenz gauge.]

Basic structure of the inhomogenous wave equation:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t) \qquad (6.32)$$

Solution of (6.32) with outgoing-wave b.c.:

$$\psi(\mathbf{x},t) = \int\int G^+(\mathbf{x},t,\mathbf{x}',t')f(\mathbf{x}',t')d^3x'dt' \qquad (6.45')$$

where $G^+(\mathbf{x},t,\mathbf{x}',t') = \dfrac{\delta\left[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|}$ (6.44)

$f(\mathbf{x}',t')$ in (6.45) is evaluated at the retarded time.

is the solution of

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G^+(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \qquad (6.41)$$

with outgoing wave b.c.

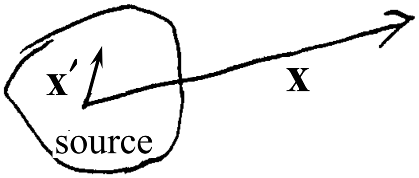
9.1 Radiation of a Localized Oscillating Source (*continued*)

Using (6.45′) on (6.15) & (6.16), we obtain the gereral solutions for **A** and **Φ**, which are valid for arbitrary **J** and ρ.

$$\left\{\begin{array}{l}\mathbf{A}(\mathbf{x},t)\\ \Phi(\mathbf{x},t)\end{array}\right\} = \frac{1}{4\pi}\int d^3x' \int dt' \frac{\delta\left[t'-(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c})\right]}{|\mathbf{x}-\mathbf{x}'|}\left\{\begin{array}{l}\mu_0 \mathbf{J}(\mathbf{x}',t')\\ \rho(\mathbf{x}',t')/\varepsilon_0\end{array}\right\} \qquad (6.48),(9.2)$$

In general, the sources, $\mathbf{J}(\mathbf{x}',t')$ and $\rho(\mathbf{x}',t')$, contain a static part and a time dependent part. For static $\mathbf{J}(\mathbf{x})$ and $\rho(\mathbf{x})$, (9.2) gives the static **A** and **Φ** in Ch. 5 and Ch. 1, respecticely.

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \qquad (5.32)$$



$$\Phi(\mathbf{x}) = \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \qquad (1.17)$$

Question: It is stated on p. 408 that (9.2) is valid provided no boundary surfaces are present. Why? [See discussion below (6.47) in Ch. 6 of lectures notes.]

9.1 Radiation of a Localized Oscillating Source (*continued*)

Fields by Harmonic Sources: Only time-dependent sources can radiate. Radiation from moving charges are treated in Ch. 13 and Ch. 14. Here, specialize to sources of the form (as in an antenna):

$$\begin{aligned}\rho(\mathbf{x},t)&=\rho(\mathbf{x})e^{-i\omega t}\\ \mathbf{J}(\mathbf{x},t)&=\mathbf{J}(\mathbf{x})e^{-i\omega t}\end{aligned} \tag{9.1}$$

Substituting (9.1) into (9.2) and carry out the t'-integration, we obtain

$$\mathbf{A}(\mathbf{x},t)=\mathbf{A}(\mathbf{x})e^{-i\omega t} \text{ with } \mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \tag{9.3}$$

where $k\equiv\frac{\omega}{c}$.

We shall assume that $\mathbf{J}(\mathbf{x})$ is independent of $\mathbf{A}(\mathbf{x})$, i.e., the source will not be affected by the fields they radiate. Otherwise, (9.3) is an integral equation for $\mathbf{A}(\mathbf{x})$.

9.1 Radiation of a Localized Oscillating Source (*continued*)

A simpler derivation of (9.3): We specialize to harmonic sources from the outset. Then, only (6.16) is required.

$$\nabla^2 \mathbf{A}(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{x},t) = -\mu_0 \mathbf{J}(\mathbf{x},t) \tag{6.16}$$

Let $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x})e^{-i\omega t}$ and $\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$

$\Rightarrow \left(\nabla^2 + k^2\right)\mathbf{A}(\mathbf{x}) = -\mu_0 \mathbf{J}(\mathbf{x})$ [inhomogeneous Helmholtz wave eq.]

The Green equation for the above equation is

$$\left(\nabla^2 + k^2\right) G_k(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{6.36}$$

Solution of (6.36) with outgoing wave b.c.

$$G_k(\mathbf{x},\mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \tag{6.40}$$

$$\Rightarrow \mathbf{A}(\mathbf{x}) = \int d^3x' G_k(\mathbf{x},\mathbf{x}')\frac{\mu_0}{4\pi}\mathbf{J}(\mathbf{x}') = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'),$$

which is (9.3).

9.1 Radiation of a Localized Oscillating Source (*continued*)

Rewrite (9.3), $$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \quad (9.3)$$

Maxwell eqs. give $$\begin{cases}\mathbf{H}=\frac{1}{\mu_0}\nabla\times\mathbf{A} & \text{(everywhere)} \quad (9.4)\\ \mathbf{E}=\frac{iZ_0}{k}\nabla\times\mathbf{H} & \text{(outside the source)} \quad (9.5)\end{cases}$$

where $Z_0=\sqrt{\mu_0/\varepsilon_0}=377\ \Omega$ (impedance of free space, p. 297).

Thus, given the source function $\mathbf{J}(\mathbf{x})$, we may in principle evaluate $\mathbf{A}(\mathbf{x})$ from (9.3) and then obtain the fields $\mathbf{H}$ and $\mathbf{E}$ from (9.4) and (9.5).

Note that $e^{-i\omega t}$ dependence has been assumed for $\mathbf{J}$, hence for all other quantities which are expressed in terms of $\mathbf{J}$.

Note: The charge distribution ρ and scalar potential Φ are not required for the determination of $\mathbf{H}$ and $\mathbf{E}$? (why?)

Show that Eq. 9.5 is valid outside the source. $\mathbf{E}=\frac{iZ_0}{k}\nabla\times\mathbf{H}$

$$\mathbf{E}(\mathbf{x},t)=-\nabla\Phi(\mathbf{x},t)-\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} \qquad (6.9)$$

Note that $e^{-i\omega t}$ dependence has been assumed for $\mathbf{A}$, Φ, $\mathbf{E}$, and $\mathbf{H}$.

Phasor $\mathbf{E}(\mathbf{x})=-\nabla\Phi(\mathbf{x})-(-i\omega)\mathbf{A}(\mathbf{x})$

$$\begin{cases}\text{Lorentz gauge} & \nabla\cdot\mathbf{A}(\mathbf{x})=\frac{i\omega}{c^2}\Phi(\mathbf{x}) \qquad (6.14')\\ \text{Homogeneous wave eq.} & \nabla^2\mathbf{A}(\mathbf{x})=i\omega\frac{i\omega}{c^2}\mathbf{A}(\mathbf{x}) \qquad (6.16')\end{cases}$$

(*outside the source*)

$$\mathbf{E}(\mathbf{x})=-\nabla\Phi(\mathbf{x})-(-i\omega)\mathbf{A}(\mathbf{x})=\frac{ic^2}{\omega}\left[\nabla\left(\frac{i\omega}{c^2}\Phi\right)-i\omega\frac{i\omega}{c^2}\mathbf{A}\right]=\underline{\frac{ic^2}{\omega}\left[\nabla(\nabla\cdot\mathbf{A}(\mathbf{x}))-\nabla^2\mathbf{A}(\mathbf{x})\right]}_{\#1}$$

$$\mathbf{E}(\mathbf{x})=\frac{iZ_0}{k}\nabla\times\mathbf{H}(\mathbf{x})=\frac{i\mu_0c^2}{\omega}\frac{1}{\mu_0}\nabla\times(\nabla\times\mathbf{A}(\mathbf{x}))=\underline{\frac{ic^2}{\omega}\left[\nabla(\nabla\cdot\mathbf{A}(\mathbf{x}))-\nabla^2\mathbf{A}(\mathbf{x})\right]}_{\#2}$$

Since $\#1=\#2 \;\Rightarrow\; \mathbf{E}(\mathbf{x})=\frac{iZ_0}{k}\nabla\times\mathbf{H}(\mathbf{x})$ is valid outside the source.

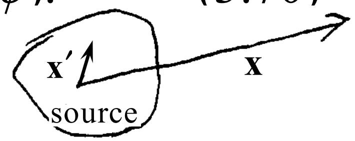
9.1 Radiation of a Localized Oscillating Source (*continued*)

Near-Field Expansion of $\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')$ (9.3)

Before going into algebraic details, we may readily observe some general properties of $\mathbf{A}(\mathbf{x})$ near the source ($r \ll \lambda$).

For $\mathbf{x}$ outside the source and $r \ll \lambda$ (or $kr \ll 1$), we let $e^{ik|\mathbf{x}-\mathbf{x}'|}\approx 1$ and use

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi). \qquad (3.70)$$



Since $r > r'$, we have $r_> = r$ and $r_< = r'$.

$$\Rightarrow \mathbf{A}(\mathbf{x}) \underset{\substack{\uparrow \\ \boxed{kr \ll 1}}}{\approx} \mu_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{1}{r^{l+1}}Y_{lm}(\theta,\phi)\int d^3x'\mathbf{J}(\mathbf{x}')r'^{l}Y_{lm}^*(\theta',\phi') \qquad (9.6)$$

The integral in (9.6) yields multipole coefficients as in (4.2). Thus, (9.6) shows that, for $kr \ll 1$, $\mathbf{A}(\mathbf{x})$ can be decomposed into multipole fields, which fall off as $r^{-(l+1)}$ just as the static multipole fields, but with the $e^{-i\omega t}$ dependence. However, we will show later that, far from the source ($kr \gg 1$), $\mathbf{A}(\mathbf{x})$ behaves as an outgoing spherical wave.

9.1 Radiation of a Localized Oscillating Source (*continued*)

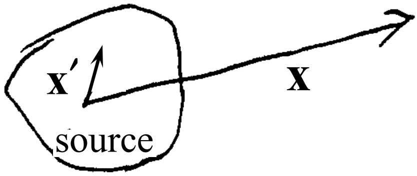
Full Expansion of $\mathbf{A}(\mathbf{x})$: We may in fact expand $\mathbf{A}(\mathbf{x})$, without approximations, by using (9.98). For $\mathbf{x}$ outside the source, we have $r_> = |\mathbf{x}| = r,\ r_< = |\mathbf{x}'| = r'$. Hence, (9.98) can be written

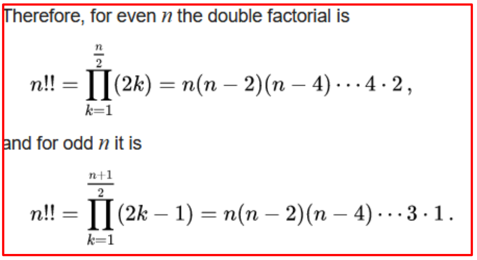
$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr')h_l^{(1)}(kr) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Substituting this equation into $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')$, we obtain

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr)Y_{lm}(\theta,\phi)\int d^3x'\mathbf{J}(\mathbf{x}')j_l(kr')Y_{lm}^*(\theta',\phi'), \quad (9.11)$$

where $h_l^{(1)}(kr) = \frac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}} \sum_{n=0}^{l} a_n (ikr)^n$




Therefore, for even n the double factorial is

$$n!! = \prod_{k=1}^{\frac{n}{2}}(2k) = n(n-2)(n-4)\cdots 4\cdot 2\,,$$

and for odd n it is

$$n!! = \prod_{k=1}^{\frac{n+1}{2}}(2k-1) = n(n-2)(n-4)\cdots 3\cdot 1\,.$$

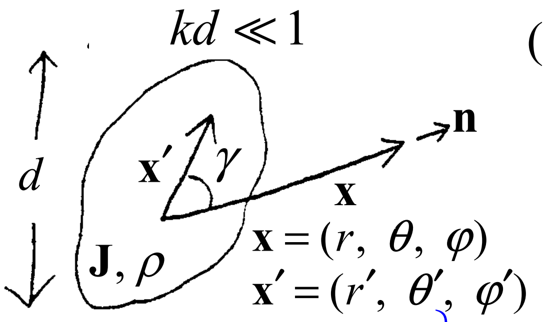
with $a_n = \frac{(-1)^n(2l-n)!}{(2l-1)!!(2l-2n)!!n!}$ $(a_0 = 1,\ a_1 = -1, \cdots)$

(See Abramowitz & Stegun, "Handbook of Mathematical Functions," p. 439.)

double factorial/semifactorial : $(2l-1)!! = (2l-1)(2l-3)\cdots 3\cdot 1$

(9.11) is an exact expression for $\mathbf{A}(\mathbf{x})$. We now assume $kd \ll 1$ (i.e., source dimension ≪ wavelength). Then, $kr' \ll 1$ and $j_l(kr')$ reduces to

$$j_l(kr')\big|_{kr' \ll 1} = \frac{1}{(2l+1)!!}(kr')^l \qquad (9.88)$$



Substituting $h_l^{(1)}(kr) = \frac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}} \sum_{n=0}^{l} a_n (ikr)^n$

and (9.88) into (9.11), we obtain

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \left\{ \begin{array}{l} \frac{1}{2l+1} Y_{lm}(\theta,\phi) \frac{e^{ikr}}{r^{l+1}} [1 + a_1(ikr) + a_2(ikr)^2 + \cdots + a_l(ikr)^l] \\ \cdot \int d^3x' \mathbf{J}(\mathbf{x}') r'^l Y_{lm}^*(\theta',\phi') \end{array} \right\} \quad (1)$$

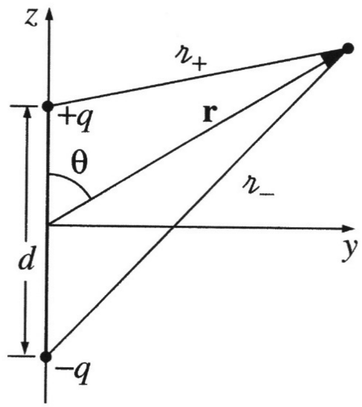
(1) is the combination of (9.6) and (9.12) in Jackson. It is valid for $kd \ll 1$ and any $\mathbf{x}$ outside the source. The region outside the source is commonly divided into 3 zones (by their different physical characters):

The near (static) zone: $d \ll r \ll \lambda$ $(\Rightarrow kr \ll 1)$
The intermediate (induction) zone: $d \ll r \sim \lambda$ $(\Rightarrow kr \sim 1)$
The far (radiation) zone: $d \ll \lambda \ll r$ $(\Rightarrow kr \gg 1)$

Griffiths

11.1.2 Electric Dipole Radiation

Consider two point charges of $+q$ and $-q$ separating by a distance $d(t)$. Assume $d(t)$ can be expressed in sinusoidal form.



The result is an oscillating electric dipole:

$$\mathbf{p}(t) = qd(t)\hat{\mathbf{z}} = qd\cos(\omega t)\hat{\mathbf{z}} = p_0\cos(\omega t)\hat{\mathbf{z}}, \quad \text{where } p_0 \equiv qd.$$

The retarded potential is:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}$$

Griffiths

Electric Dipole Radiation: Approximations

Approximation #1: Make this physical dipole into a perfect dipole.

$$d \ll r$$

Estimate the spearation distances by the law of cosines.

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2} \cong r(1 \mp \frac{d}{2r}\cos\theta)$$

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}(1 \pm \frac{d}{2r}\cos\theta)$$

$$\cos[\omega(t - \mathscr{r}_\pm / c)] \cong \cos[\omega(t - \frac{r}{c}) \pm \frac{\omega d}{2c}\cos\theta]$$

$$= \cos[\omega(t - \frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta) \mp \sin[\omega(t - \frac{r}{c})]\sin(\frac{\omega d}{2c}\cos\theta)$$

Griffiths

The Retarded Scalar Potential

Approximation #2: The wavelength is much longer than the dipole size.

$$d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi}$$

$$\cos[\omega(t - \mathscr{r}_{\pm} / c)] \cong \cos[\omega(t - \frac{r}{c})]\underbrace{\cos(\frac{\omega d}{2c}\cos\theta)}_{=1} \mp \sin[\omega(t - \frac{r}{c})]\underbrace{\sin(\frac{\omega d}{2c}\cos\theta)}_{\frac{\omega d}{2c}\cos\theta}$$

$$= \cos[\omega(t - \frac{r}{c})] \mp \sin[\omega(t - \frac{r}{c})]\frac{\omega d}{2c}\cos\theta$$

The retarded scalar potential is:

$$V(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\left\{\begin{array}{l} \left[\cos[\omega(t - \frac{r}{c})] - \sin[\omega(t - \frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1 + \frac{d}{2r}\cos\theta) \\ -\left[\cos[\omega(t - \frac{r}{c})] + \sin[\omega(t - \frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1 - \frac{d}{2r}\cos\theta) \end{array}\right\}$$

$$\cong \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t - \frac{r}{c})] + \frac{1}{r}\cos[\omega(t - \frac{r}{c})]\right]$$

Griffiths

The Retarded Scalar Potential

Approximation #3: at the radiation zone. $\frac{c}{\omega} \ll r$

The retarded scalar potential is:

$$V(\mathbf{r},t) \cong \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})\right]$$

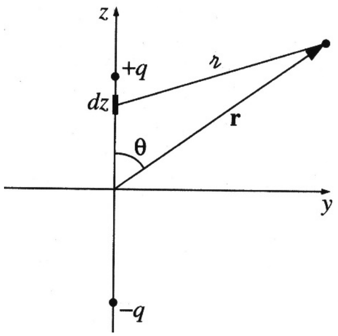
Three approximations

$$d \ll r \qquad d \ll \frac{c}{\omega}(=\frac{\lambda}{2\pi}) \qquad \frac{c}{\omega} \ll r$$

$$\Rightarrow d \ll \lambda \ll r$$

Griffiths

The Retarded Vector Potential



The retarded vector potential is determined by the current density.

$$I(t)=\frac{dq}{dt}\hat{\mathbf{z}}=-q_0\omega\sin\omega t\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'=\frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q\omega\sin[\omega(t-\mathscr{r}/c)]\hat{\mathbf{z}}}{\mathscr{r}}dz$$

$$\cong-\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]\hat{\mathbf{z}} \qquad @\,d\ll\lambda\ll r$$

Retarded potentials:

$$\begin{cases}V(\mathbf{r},t)=-\dfrac{p_0\omega}{4\pi\varepsilon_0 c}\left[\dfrac{\cos\theta}{r}\sin[\omega(t-\dfrac{r}{c})\right]\\[2ex] \mathbf{A}(\mathbf{r},t)=-\dfrac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\dfrac{r}{c})]\hat{\mathbf{z}}\end{cases}$$

$$\begin{cases}\mathbf{E}=-\nabla V-\dfrac{\partial\mathbf{A}}{\partial t}\\[2ex] \mathbf{B}=\nabla\times\mathbf{A}\end{cases}$$

Griffiths

The Electromagnetic Fields and Poynting Vector

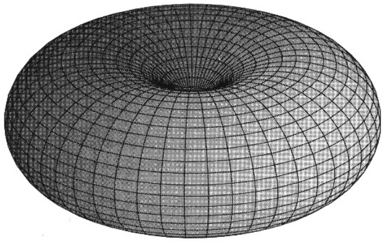
$$\begin{cases} \mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t} = -\dfrac{\mu_0 p_0 \omega^2}{4\pi\varepsilon_0 c}\left(\dfrac{\sin\theta}{r}\right)\cos[\omega(t-\dfrac{r}{c})]\hat{\boldsymbol{\theta}} \\ \mathbf{B} = \nabla\times\mathbf{A} = -\dfrac{\mu_0 p_0 \omega^2}{4\pi c}\left(\dfrac{\sin\theta}{r}\right)\cos[\omega(t-\dfrac{r}{c})]\hat{\boldsymbol{\varphi}} \end{cases}$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]\right\}^2 \hat{\mathbf{r}}$$

The total power radiated is

$$< P >= \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\left(\frac{\sin\theta}{r}\right)^2 r^2 \sin\theta d\theta d\phi$$

$$= \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$



9.2 Electric Dipole Fields and Radiation

Rewrite (1):

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \left\{ \begin{array}{l} \frac{1}{2l+1} Y_{lm}(\theta,\phi)\frac{e^{ikr}}{r^{l+1}}[1+a_1(ikr)+a_2(ikr)^2+\cdots+a_l(ikr)^l] \\ \cdot\int d^3x'\mathbf{J}(\mathbf{x}')r'^l Y^*_{lm}(\theta',\phi') \end{array} \right\} \quad (1)$$

Take the $l=0$ term $[Y_{00}=\frac{1}{\sqrt{4\pi}}]$ and denote it by $\mathbf{A}^p(\mathbf{x})$

$$\mathbf{A}^p(\mathbf{x}) = \mathbf{A}(\mathbf{x})^{l=0} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}(\mathbf{x}')$$

$$= -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}, \quad (9.16)$$

where $\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x'$ (4.8)

(9.16) gives the electric dipole contribution to the solution. It is valid for $kd \ll 1$ and any $\mathbf{x}$ outside the source.

Question: Why is there no monopole term (see p. 410)?

$$\iiint J_x dxdydz$$

$$= \iint dydz\left[\cancel{xJ_x\Big|_{-d}^{d}} - \int x\frac{\partial J_x}{\partial x}dx\right]$$

$$= -\iiint x\Big(\frac{\partial J_x}{\partial x} + \underbrace{\frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}}\Big)dxdydz$$

give no contribution because $\mathbf{J}$ is localized: $\int \frac{\partial J_y}{\partial y}dy = J_y\Big|_{-d}^{d} = 0$

$$= -\int x\nabla\cdot\mathbf{J}d^3x$$

$$\Rightarrow \int \mathbf{J}d^3x = -\int \mathbf{x}\nabla\cdot\mathbf{J}d^3x$$

$$= -i\omega\underbrace{\int \mathbf{x}\rho(\mathbf{x})d^3x}_{\mathbf{p}} = -i\omega\mathbf{p}$$

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0$$

9.2 Electric Dipole Fields and Radiation (*continued*)

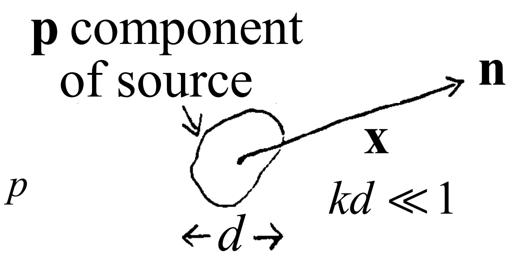
Rewrite (9.16): $\mathbf{A}^p(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}$ (9.16)

From (9.4), $\mathbf{H}^p = \frac{1}{\mu_0}\nabla\times\mathbf{A}^p$ and from (9.5), $\mathbf{E}^p = \frac{iZ_0}{k}\nabla\times\mathbf{H}^p$

$$\Rightarrow \begin{cases} \mathbf{H}^p = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ \mathbf{E}^p = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \end{cases} \quad (9.18)$$

In the far zone ($kr \gg 1$), (9.18) reduces to a spherical wave

$$\begin{cases} \mathbf{H}^p \simeq \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r} \\ \mathbf{E}^p \simeq Z_0\mathbf{H}^p\times\mathbf{n} \end{cases} \quad (9.19)$$



In (9.19), we see that $\mathbf{E}^p$ and $\mathbf{H}^p$ are in phase, and $\mathbf{E}^p$, $\mathbf{H}^p$, and $\mathbf{n}$ are mutually perpendicular. This is a general property of EM waves in unbounded, uniform space. Given any two of these quantities, we can find the third.

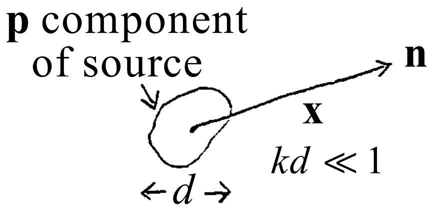
9.2 Electric Dipole Fields and Radiation (*continued*)

$$\begin{cases} \mathbf{H}^p = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ \mathbf{E}^p = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r}+[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \end{cases} \quad (9.18)$$

$$\begin{aligned} \mathbf{H}^p &\approx \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r}(1+ikr+...)\left(1-\frac{1}{ikr}\right) \\ &\approx \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r}(1+ikr-\frac{1}{ikr}-1) \\ &\approx \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r}(-\frac{1}{ikr}) = \frac{ck}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{i}{r^2} \end{aligned}$$

In the near zone ($kr \ll 1$), (9.18) reduces to

$$\begin{cases} \mathbf{H}^p \simeq \frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r^2} \\ \mathbf{E}^p \simeq \frac{1}{4\pi\varepsilon_0}[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\frac{1}{r^3} \end{cases} \quad (9.20)$$



$$\Rightarrow \begin{cases} \text{(i) } \mathbf{E}^p \text{ and } \mathbf{H}^p \text{ are } 90^\circ \text{ out of phase} \Rightarrow \text{average power} = 0. \\ \text{(ii) } \mathbf{E}^p \text{ has the same spatial pattern as that of the static electric} \\ \quad\quad \text{dipole in (4.13), but with } e^{-i\omega t} \text{ dependence.} \\ \text{(iii) } \mu_0|H|^2 \sim (kr)^2\varepsilon_0|E|^2 \Rightarrow \mathbf{E}\text{-field energy} \gg \mathbf{B}\text{-field energy.} \end{cases}$$

Questions: (i) Why does $\mathbf{E}^p$ have the static field pattern?

(ii) To obtain (9.20), we have neglected a few terms in (9.18). But some of the neglected terms are still important in the near zone? What are they and in what sense are they important?

9.2 Electric Dipole Fields and Radiation (*continued*)

$$\boxed{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}}$$
$$\boxed{(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}\times\mathbf{b})}$$

$\left\langle \frac{dP}{d\Omega}\right\rangle_t$ = time-averaged power in the far zone/unit solid angle

$\boxed{(9.19)}$

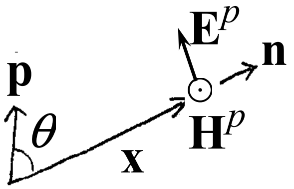
$$= \frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot\left(\mathbf{E}^p\times\mathbf{H}^{p*}\right)\right] \quad (9.21)$$

$$= \frac{c^2 Z_0}{32\pi^2}k^4\,|\underbrace{(\mathbf{n}\times\mathbf{p})\times\mathbf{n}}|^2 \qquad (9.22)$$

$$\boxed{\begin{aligned} r^2\mathbf{n}\cdot\left(\mathbf{E}^p\times\mathbf{H}^{p*}\right) &= r^2\mathbf{n}\cdot\left(Z_0\mathbf{H}^p\times\mathbf{n}\times\mathbf{H}^{p*}\right)\\ &= Z_0 r^2\mathbf{n}\cdot\left(\tfrac{ck^2}{4\pi}\tfrac{e^{ikr}}{r}\tfrac{ck^2}{4\pi}\tfrac{e^{-ikr}}{r}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\times(\mathbf{n}\times\mathbf{p})]\right)\\ &= Z_0\tfrac{ck^2}{4\pi}\tfrac{ck^2}{4\pi}\mathbf{n}\cdot[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\times(\mathbf{n}\times\mathbf{p})]\end{aligned}}$$

This vector gives the direction of $\mathbf{E}^p$, i.e., the polarization of the radiation (see figure below.)

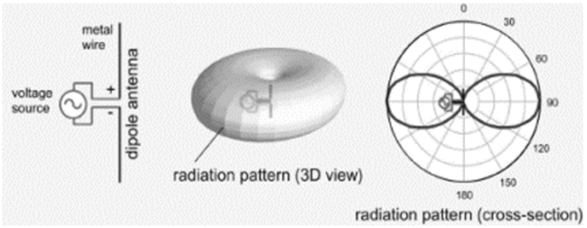
$$\Rightarrow \langle P\rangle_t = \text{total power radiated} = \frac{c^2 Z_0 k^4}{12\pi}|\mathbf{p}|^2 \qquad (9.24)$$



In general, $\mathbf{p}=p_x e^{i\alpha}\mathbf{e}_x+p_y e^{i\beta}\mathbf{e}_y+p_z e^{i\gamma}\mathbf{e}_z$. If $\alpha=\beta=\gamma$, then $\mathbf{p}$ has a fixed direction, $\mathbf{p}=\mathbf{p}_0 e^{i\alpha}$ with $\mathbf{p}_0=p_x\mathbf{e}_x+p_y\mathbf{e}_y+p_z\mathbf{e}_z$, and

$$\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{c^2 Z_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta. \qquad (9.23)$$

Otherwise, the direction of $\mathbf{p}$ (hence $\left\langle \frac{dP}{d\Omega}\right\rangle_t$) vary with time, but $\langle P\rangle_t$ is still given by (9.24).



dipole radiation pattern

9.3 Magnetic Dipole and Electric Quadrupole Field

Rewrite (1):

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \left\{ \begin{array}{l} \frac{1}{2l+1} Y_{lm}(\theta,\phi)\frac{e^{ikr}}{r^{l+1}}[1+a_1(ikr)+a_2(ikr)^2+\cdots+a_l(ikr)^l] \\ \cdot\int d^3x'\mathbf{J}(\mathbf{x}')r'^l Y^*_{lm}(\theta',\phi') \end{array}\right\} \quad (1)$$

Take the $l=1$ terms $[a_1=-1]$

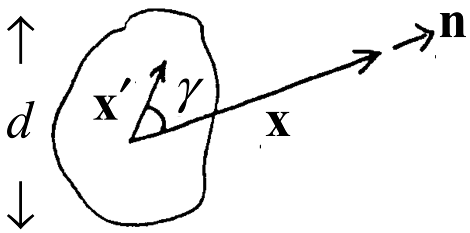
$$\mathbf{A}(\mathbf{x})^{l=1} = \frac{\mu_0}{3}\frac{e^{ikr}}{r^2}(1-ikr)\underbrace{\sum_{m=-1,0,1} Y_{1m}(\theta,\phi)\int d^3x'\mathbf{J}(\mathbf{x}')r' Y^*_{1m}(\theta',\phi')}$$

$$\sum_{m=-1,0,1} Y_{1m}(\theta,\phi)Y^*_{1m}(\theta',\phi') \overset{\text{p. 109}}{=} \tfrac{3}{8\pi}\sin\theta\sin\theta' e^{i(\phi-\phi')}$$

$$+\tfrac{3}{4\pi}\cos\theta\cos\theta' + \tfrac{3}{8\pi}\sin\theta\sin\theta' e^{-i(\phi-\phi')}$$

$$= \tfrac{3}{4\pi}[\sin\theta\sin\theta'\cos(\phi-\phi')+\cos\theta\cos\theta']$$

$$\underset{\text{set } l=1 \text{ in (3.68)}}{=} \tfrac{3}{4\pi}\cos\gamma = \tfrac{3}{4\pi r'}\mathbf{n}\cdot\mathbf{x}'$$



$$(\mathbf{x}'\times\mathbf{J})\times\mathbf{n}=\mathbf{J}(\mathbf{n}\cdot\mathbf{x}')-\mathbf{x}'(\mathbf{n}\cdot\mathbf{J})$$

$$\tfrac{1}{2}(\mathbf{x}'\times\mathbf{J})\times\mathbf{n}=-\tfrac{1}{2}\big(\mathbf{J}(\mathbf{n}\cdot\mathbf{x}')+\mathbf{x}'(\mathbf{n}\cdot\mathbf{J})\big)+\mathbf{J}(\mathbf{n}\cdot\mathbf{x}')$$

Thus,

$$\mathbf{A}(\mathbf{x})^{l=1}=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(\tfrac{1}{r}-ik)\int d^3x'\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}') \qquad (9.30)$$

$$=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(\tfrac{1}{r}-ik)\Big\{\underbrace{\int d^3x'\tfrac{1}{2}[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J}+(\mathbf{n}\cdot\mathbf{J})\mathbf{x}']}_{\text{electric quadrupole radiation}}+\underbrace{\int d^3x'\tfrac{1}{2}(\mathbf{x}'\times\mathbf{J})\times\mathbf{n}}_{\text{magnetic dipole radiation}}\Big\}$$

$$=\mathbf{A}^Q+\mathbf{A}^m,$$

where $\mathbf{A}^m(\mathbf{x})=\frac{ik\mu_0}{4\pi}(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$ $\left[\begin{array}{l}\text{for } kd\ll 1 \text{ and any}\\ \mathbf{x} \text{ outside the source}\end{array}\right]$ (9.33)

with $\mathbf{m}=\frac{1}{2}\int(\mathbf{x}\times\mathbf{J})d^3x$ [magnetic dipole moment]. $\mathbf{A}^m$ gives the magnetic dipole contribution through (9.4) and (9.5) (see p.15):

$$\begin{cases}\mathbf{H}^m=\frac{1}{4\pi}\left\{k^2(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{e^{ikr}}{r}+[3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} & (9.35)\\ \mathbf{E}^m=-\frac{Z_0}{4\pi}k^2(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) & (9.36)\end{cases}$$

9.3 Magnetic Dipole and Electric Quadrupole Fields (*continued*)

In the far zone ($kr \gg 1$), we have the spherical wave sloution:

$$\begin{cases} \mathbf{H}^m \simeq \frac{k^2}{4\pi}(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\,\frac{e^{ikr}}{r} \\ \mathbf{E}^m \simeq Z_0\mathbf{H}^m\times\mathbf{n} \end{cases} \Rightarrow \begin{cases} \left\langle \frac{dP}{d\Omega} \right\rangle_t \simeq \frac{Z_0}{32\pi^2}k^4\,|\underbrace{\mathbf{m}\times\mathbf{n}}_{\boxed{\Rightarrow \text{direction of } \mathbf{E}^m}}|^2 \\ \langle P \rangle_t \simeq \frac{Z_0}{12\pi}k^4\,|\mathbf{m}|^2 \end{cases}$$

In the near zone ($kr \ll 1$),

$$\begin{cases} \mathbf{H}^m \simeq \frac{1}{4\pi}[3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}]\frac{1}{r^3} \\ \mathbf{E}^m \simeq \frac{Z_0 k}{4\pi i}(\mathbf{n}\times\mathbf{m})\frac{1}{r^2} \end{cases} \Rightarrow$$

(i) $\mathbf{E}^m$ and $\mathbf{H}^m$ are 90° out of phase

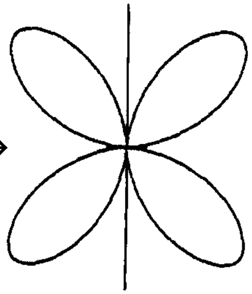
$\Rightarrow$ average power $= 0$.

(ii) $\mathbf{H}^m$ has the same spatial pattern as that of the static magnetic dipole in (5.56), but with $e^{-i\omega t}$ dependence.

(iii) **B**-field energy $\gg$ **E**-field energy.

The electric quadrupole radiation, discussed in (9.37)-(9.52), is more complicated. Here, we only illustrate its radiation pattern by the figure to the right.

quadrupole radiation pattern →



Comparison between Static and Time-dependent Cases

	relations between ρ, **J**, **E**, and **B**	multipole expansion	definition of multipole moments	r-dependence of **E** and **B** (d : dimension of the source)
static case	$\rho(\mathbf{x}) \leftrightarrow \mathbf{E}(\mathbf{x})$ $\mathbf{J}(\mathbf{x}) \leftrightarrow \mathbf{B}(\mathbf{x})$	spherical harmonics expansion [(3.70)] or Taylor series [(4.10)] of $\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$q = \int \rho(\mathbf{x}')d^3x'$ $\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x'$ $Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')d^3x'$ $\mathbf{m} = \frac{1}{2}\int \mathbf{x}'\times\mathbf{J}(\mathbf{x}')d^3x'$	**E** or **B** $\propto 1/r^{l+2}$ For $r \sim d$, all multipole fields can be significant. For $r \gg d$, multipole fields are dominated by the lowest-order nonvanishing term.
time-dependent case	$\begin{Bmatrix}\rho(\mathbf{x})\\ \updownarrow \\ \mathbf{J}(\mathbf{x})\end{Bmatrix} \leftrightarrow \begin{Bmatrix}\mathbf{E}(\mathbf{x})\\ \updownarrow \\ \mathbf{B}(\mathbf{x})\end{Bmatrix}$ $\Rightarrow$ EM waves	spherical harmonics expansion [(9.98)] of $\frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	There is no time-dependent monopole for an isolated source (see p. 410). **p**, Q_{ij}, and **m** have the same expressions as those of their static counterparts, but with the $e^{-i\omega t}$ time dependence. In time-dependent cases, electric multipoles can generate **B**-fields and magnetic multipoles can generate **E**-fields.	(a) near zone $\lambda \gg r \gg d$ **E** or **B** $\propto e^{-i\omega t}/r^{l+2}$ Approx. the same field pattern and r-dependence as for the corresponding static multipole, but with $e^{-i\omega t}$ dependence (hence called quasi-static fields.) (b) far zone $r \gg \lambda \gg d$ $\mathbf{E},\mathbf{B} \propto e^{ikr-i\omega t}/r$ (spherical EM waves) All multipole fields $\propto 1/r$, relative power levels unchanged with distance.

Correction to Jackson Eq. 9.8

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\,\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}') \qquad (9.3)$$

$$|\mathbf{x}-\mathbf{x}'|\simeq r-\mathbf{n}\cdot\mathbf{x}'=r\Big(1-\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\Big)$$

n: a unit vector in the direction of **x** (9.7)

$$\lim_{kr\to\infty}\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,e^{-ik\mathbf{n}\cdot\mathbf{x}'}\mathbf{J}(\mathbf{x}')\left[1+\frac{\mathbf{n}\cdot\mathbf{x}'}{r}+\Big(\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\Big)^2+\ldots\right] \qquad (9.8')$$

$$e^{-ik\mathbf{n}\cdot\mathbf{x}'}=\big(1-ik\mathbf{n}\cdot\mathbf{x}'+(-ik\mathbf{n}\cdot\mathbf{x}')^2/2+\ldots\big)=\sum_{n=0}^{\infty}\frac{(-ik\mathbf{n}\cdot\mathbf{x}')^n}{n!}$$

$$\begin{aligned}\lim_{kr\to\infty}\mathbf{A}(\mathbf{x})&=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\sum_{n=0}^{\infty}\frac{(-ik\mathbf{n}\cdot\mathbf{x}')^n}{n!}\mathbf{J}(\mathbf{x}')\left[1+\frac{\mathbf{n}\cdot\mathbf{x}'}{r}+\Big(\frac{\mathbf{n}\cdot\mathbf{x}'}{r}\Big)^2+\ldots\right]\\
&=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\mathbf{J}(\mathbf{x}')+\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\mathbf{J}(\mathbf{x}')\left[\frac{\mathbf{n}\cdot\mathbf{x}'}{r}+\frac{-ik\mathbf{n}\cdot\mathbf{x}'}{1}\right]+\ldots\\
&=\underbrace{\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\mathbf{J}(\mathbf{x}')}_{\ell=0\ (9.13)}+\underbrace{\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\Big(\frac{1}{r}-ik\Big)\int d^3x'\,\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}')}_{\ell=1\ (9.30)}+\ldots\end{aligned}$$

Induced Electric and Magnetic Dipoles

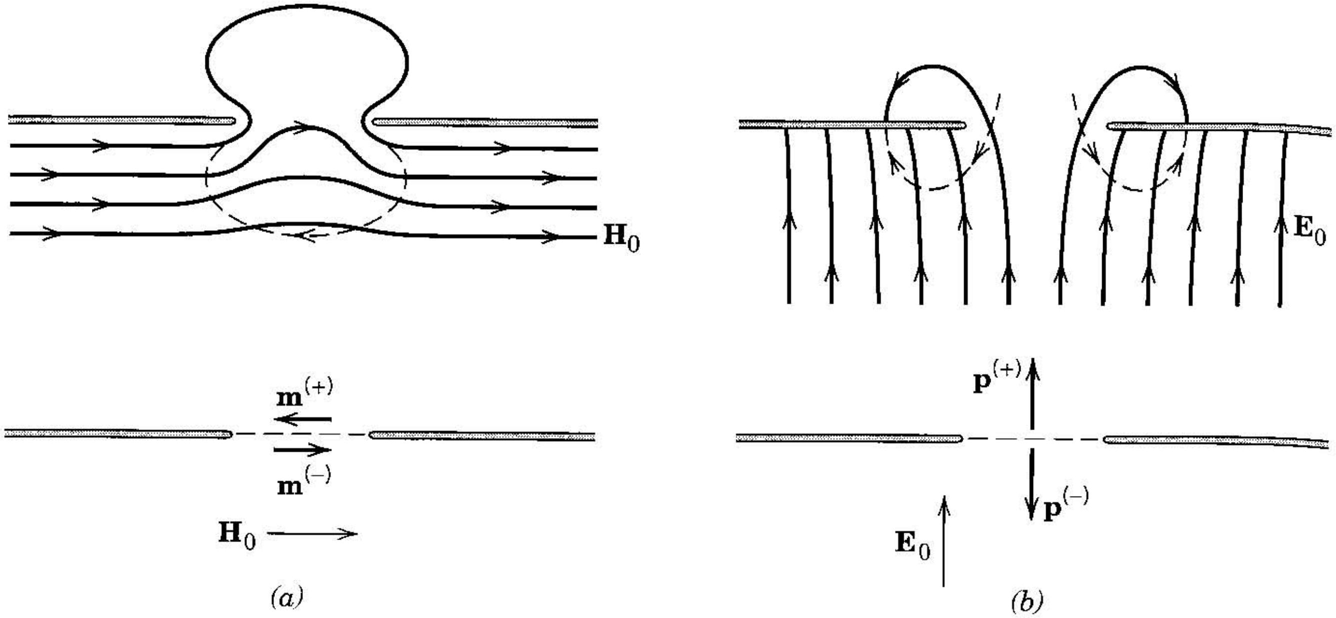


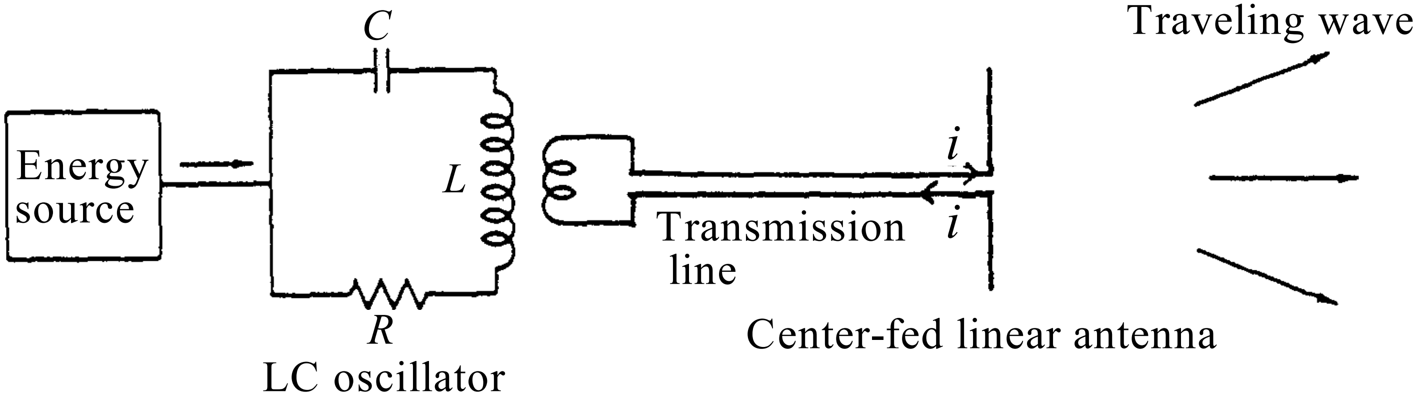
Figure 9.4 Distortion of (*a*) the tangential magnetic field and (*b*) the normal electric field by a small aperture in a perfectly conducting surface. The effective dipole moments, as viewed from above and below the surface, are indicated beneath.

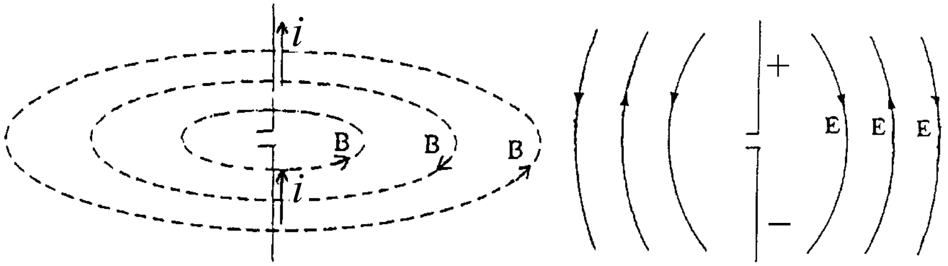
Hans Bethe published a paper, titled "Theory of diffraction by small holes" in Physical Review in 1944.

He won the 1967 Nobel Prize in Physics.

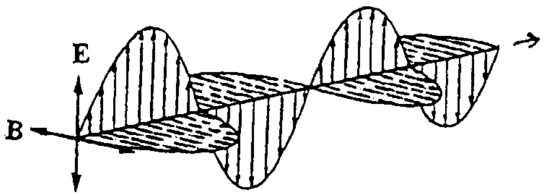
9.4 Center-Fed Linear Antenna

A Qualitative Look at the Center - Fed Linear Antenna :





near zone



far zone

In the near zone, **E** and **B** are principally generated by ρ and **J**, respectively (⇒ largely static field patterns). In the far zone, **E** and **B** are regenerative through $\frac{d}{dt}\mathbf{B}$ and $\frac{d}{dt}\mathbf{E}$ (⇒ EM waves).

9.4 Center-fed Linear Antenna (*continued*)

Detailed Analysis: The center-fed linear antenna is a case of special interest, because it allows the solution of (9.3) in a closed form for any value of kd, whereas in Secs. 9.2 and 9.3, we assume $kd \ll 1$.

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \qquad (9.3)$$

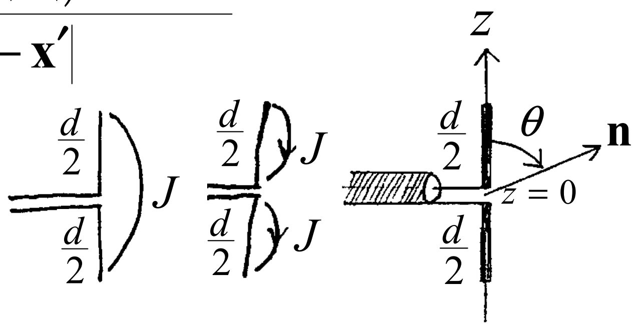
$$\text{where } \mathbf{J}(\mathbf{x}) = I\sin\left(\tfrac{kd}{2} - k|z|\right)\delta(x)\delta(y)\mathbf{e}_z \qquad (9.53)$$

reasonable assumption

$$\Rightarrow \quad \mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin\left(\tfrac{kd}{2} - k|z'|\right)e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

Note: (i) $\mathbf{J}$ is symmetric about $z = 0$. $\mathbf{J}(z) = \mathbf{J}(-z) \rightarrow$

(ii) I is the peak current only when $kd \geq \pi$.



Question: The antenna appears to be an open circuit. How can there be current flowing on it?

9.4 Center-fed Linear Antenna (*continued*)

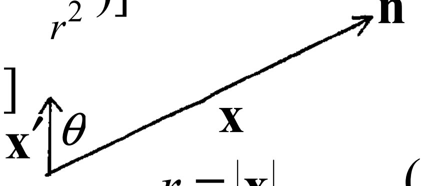
$$\boxed{\sqrt{1-x}=1-\tfrac{1}{2}x-\tfrac{1}{8}x^2-\tfrac{1}{16}x^3+\ldots}$$

$$|\mathbf{x}-\mathbf{x}'|=(r^2-2rr'\cos\theta+r'^2)^{\frac{1}{2}}=r[1-(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\tfrac{r'^2}{r^2})]^{\frac{1}{2}}$$

$$=r[1-\tfrac{1}{2}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\tfrac{r'^2}{r^2})-\tfrac{1}{8}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\tfrac{r'^2}{r^2})^2+\cdots]$$

$$=r-\mathbf{n}\cdot\mathbf{x}'+\tfrac{1}{2r}[r'^2-(\mathbf{n}\cdot\mathbf{x}')^2]+\cdots$$

$$\Rightarrow |\mathbf{x}-\mathbf{x}'|\approx r-\mathbf{n}\cdot\mathbf{x}' \text{ if } r\gg r'$$



$r=|\mathbf{x}|$ (2)

$r'=|\mathbf{x}'|$

Hence, if $r\gg d$, we can write $|\mathbf{x}-\mathbf{x}'|\simeq r-z'\cos\theta$.

$$\Rightarrow \mathbf{A}(\mathbf{x})\approx \mathbf{e}_z\frac{\mu_0 I e^{ikr}}{4\pi}\int_{-d/2}^{d/2}dz'\frac{\sin\left(\frac{kd}{2}-k|z'|\right)e^{-ikz'\cos\theta}}{\underbrace{r-z'\cos\theta}_{\approx r}} \qquad (9.54)$$

$$=\mathbf{e}_z\frac{\mu_0 I e^{ikr}}{2\pi kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \qquad (9.55)$$

Note: $z'\cos\theta$ in $\frac{1}{r-z'\cos\theta}$ can be neglected if $r\gg d$. But $z'\cos\theta$ in $e^{ik(r-z'\cos\theta)}$ makes an important contribution to the phase angle even at $r\gg d$.

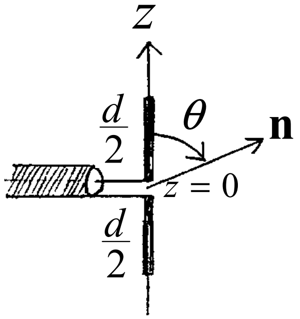
9.4 Center-fed Linear Antenna (*continued*)

(9.16 & 9.19)

In the far zone, $\boxed{\mathbf{E}=Z_0\mathbf{H}\times\mathbf{n}}$ $\boxed{\mathbf{H}=\frac{1}{\mu_0}\nabla\times\mathbf{A}=\frac{ik}{\mu_0}\mathbf{n}\times\mathbf{A}\Rightarrow|\mathbf{H}|=\frac{k\sin\theta|\mathbf{A}|}{\mu_0}}$

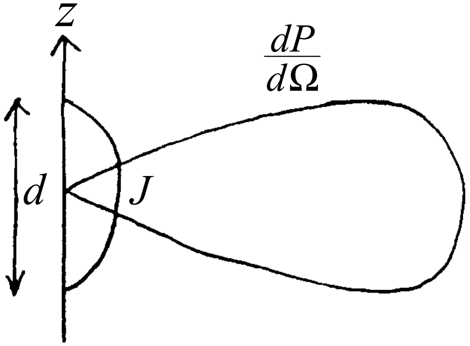
$$\left\langle\frac{dP}{d\Omega}\right\rangle_t=\frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*\right]\overset{\downarrow}{=}\frac{Z_0}{2}r^2|\mathbf{H}|^2\overset{\downarrow}{=}\frac{Z_0}{2\mu_0^2}k^2r^2\sin^2\theta|\mathbf{A}|^2 \qquad (3)$$

$$=\frac{Z_0I^2}{8\pi^2}\left|\frac{\cos(\frac{kd}{2}\cos\theta)-\cos(\frac{kd}{2})}{\sin\theta}\right|^2,\quad\left[\begin{array}{l}\text{for } r\gg d\\ \text{and any } kd\end{array}\right] \qquad (9.56)$$

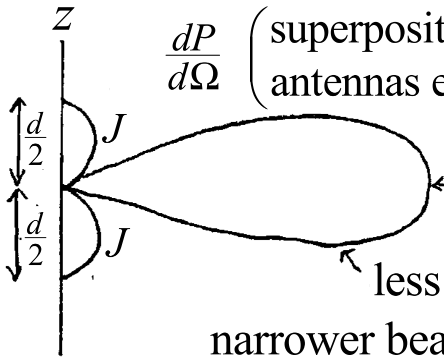


$$=\frac{Z_0I^2}{8\pi^2}\begin{cases}\cos^2(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd=\pi\\ 4\cos^4(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd=2\pi\end{cases} \qquad (9.57)$$

half-wave antenna
$(kd=\pi,\ \lambda=2d)$



full-wave antenna
$(kd=2\pi,\ \lambda=d)$



(superposition of 2 half-wave antennas excited in phase)

most coherent

less coherent

narrower beam width

Rewrite (9.56)

$$\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{Z_0 I^2}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2, \begin{bmatrix} \text{for } r \gg d \\ \text{and any } kd \end{bmatrix} \qquad (9.56)$$

Limiting case (dipole approximation): $kd \ll 1$ (i.e., $\lambda \gg d$)

$$\cos x \simeq 1 - \frac{x^2}{2} \quad (x \ll 1)$$

$$\Rightarrow \begin{cases} \cos\left(\frac{kd}{2}\cos\theta\right) \simeq 1 - \frac{k^2 d^2}{8}\cos^2\theta \\ \cos\left(\frac{kd}{2}\right) \simeq 1 - \frac{k^2 d^2}{8} \end{cases}$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle_t \approx \frac{Z_0 I^2}{8\pi^2} \left| \frac{1 - \frac{k^2 d^2}{8}\cos^2\theta - 1 + \frac{k^2 d^2}{8}}{\sin\theta} \right|^2$$

$$= \frac{Z_0 I^2}{512\pi^2}(kd)^4 \sin^2\theta \quad [\text{valid for } kd \ll 1] \qquad (4)$$

This has the same k and θ depedence as in (9.23, electric dipole), which was derived by assuming $kd \ll 1$.

9.4 Center-fed Linear Antenna (*continued*)

Radiation Resistance and Equivalent Circuit:

$$\mathbf{J}(\mathbf{x}) = I\sin\left(\tfrac{kd}{2} - k\,|\,z\,|\right)\delta(x)\delta(y)\mathbf{e}_z \approx \underbrace{\tfrac{kd}{2}I}_{I_0\,(\text{peak current, } \because |z| \le d)}\left(1 - \tfrac{2|z|}{d}\right)\delta(x)\delta(y)\mathbf{e}_z$$

$$\boxed{kd \ll 1}$$

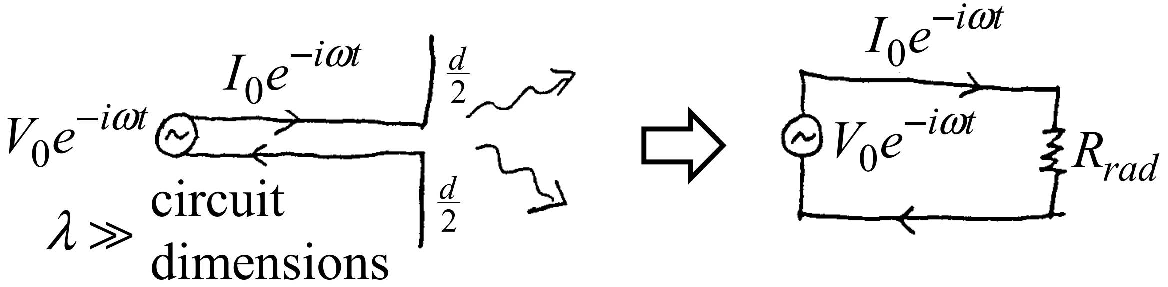
Thus, from (4), $\left\langle \frac{dP}{d\Omega} \right\rangle_t \approx \frac{Z_0 I^2}{512\pi^2}(kd)^4 \sin^2\theta = \frac{Z_0 I_0^2}{128\pi^2}(kd)^2 \sin^2\theta \quad (9.28)$

$$\Rightarrow \langle P \rangle_t \approx \int \left\langle \frac{dP}{d\Omega} \right\rangle_t d\Omega = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{Z_0 I_0^2}{48\pi}(kd)^2 \qquad (9.29)$$

$$= \frac{I_0^2}{2} R_{rad},$$

R_{rad}: radiation resistance.
R_{rad} is part of the field definition of impedance, see 2nd term in (6.137).

where $R_{rad} \equiv \frac{Z_0}{24\pi}(kd)^2 \approx 5(kd)^2$ Ohms [See pp. 412-3.]

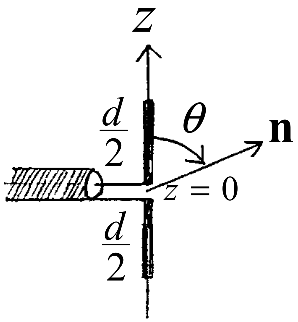


Equivalent circuit for a center-fed antenna

9.4 Center-fed Linear Antenna (*continued*)

Problems: 9.16 & 9.17

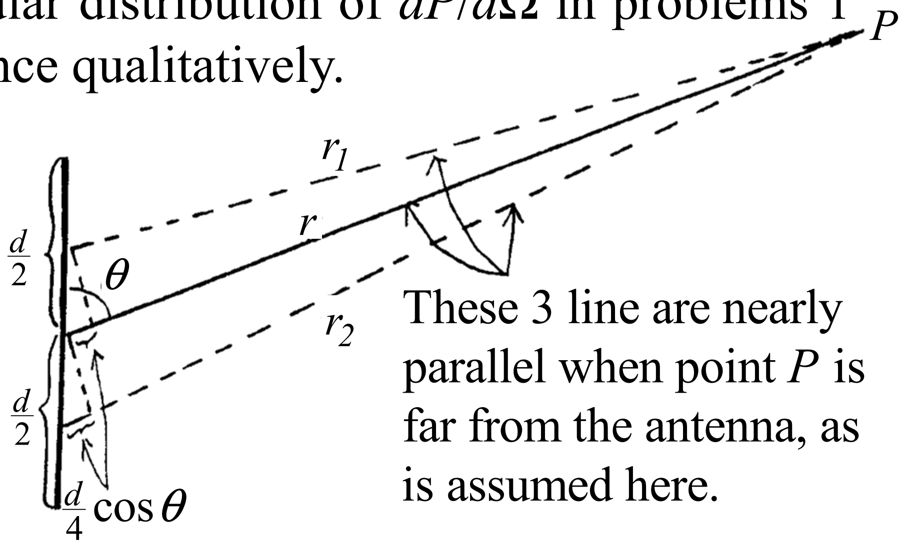
1. The full-wave antenna radiation in (9.57) can be thought of as the superposition of two half-wave antennas, one above the other, excited in phase. Demonstrate this by rederiving $dP/d\Omega$ for the full-wave antenna [(9.57), $kd = 2\pi$] by superposing the fields of two half-wave antennas (each of length $d/2$, see figure below).
2. If the two half-wave antennas in problem 1 are excited 180° out of phase, derive $dP/d\Omega$ again by the method of superposition.
3. Plot the approximate angular distribution of $dP/d\Omega$ in problems 1 and 2. Explain the difference qualitatively.



single antenna of length d

antenna 1 of length $\frac{d}{2} \rightarrow$

antenna 2 of length $\frac{d}{2} \rightarrow$

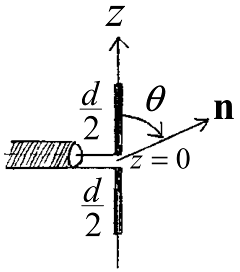


These 3 line are nearly parallel when point P is far from the antenna, as is assumed here.

9.4 Center-fed Linear Antenna (*continued*)

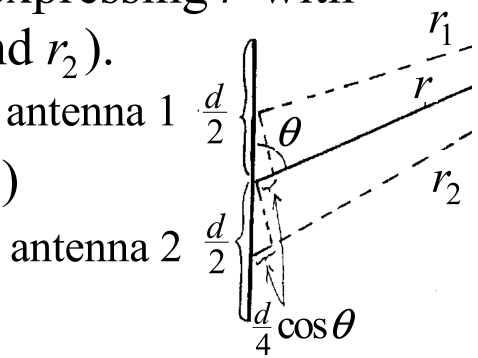
Solution to problem 1: Principle of superposition requires that we add the fields (not the powers) of the 2 antennas, each of total length $\frac{d}{2}$.

Rewrite (9.55)



$$\mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{2\pi k r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \qquad (9.55)$$

(9.55) applies to a single antenna of total length d (see fig. above.) So the field of each of the 2 antennas in this problem can be obtained from (9.55) by replacing d in (9.55) with $\frac{d}{2}$ and expressing r with respect to the center of each antenna (i.e., by r_1 and r_2).



$$\mathbf{A}_{1,2} = \mathbf{e}_z \frac{\mu_0 I e^{ikr_{1,2}}}{2\pi k r_{1,2}}\left[\frac{\cos\left(\frac{kd}{4}\cos\theta\right) - \cos\left(\frac{kd}{4}\right)}{\sin^2\theta}\right], \quad (5)$$

where $r_1 = r - \frac{d}{4}\cos\theta$ and $r_2 = r + \frac{d}{4}\cos\theta$.

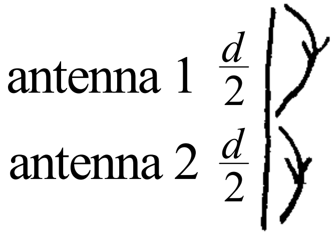
We may approximate $r_{1,2}$ in the denominator of (5) by r, but must use the correct $r_{1,2}$ for the phase angles in the exponential terms.

9.4 Center-fed Linear Antenna (*continued*)

It is assumed that each antenna in this problem is excited in the half-wave pattern, hence we set $k\frac{d}{2}=\pi$ in (5) and the superposed field of the 2 antennas (excited in phase) is given by

$$\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2=\mathbf{e}_z\frac{\mu_0}{2\pi}\frac{I}{kr}e^{ikr}[e^{-i\frac{\pi}{2}\cos\theta}+e^{i\frac{\pi}{2}\cos\theta}]\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \qquad (6)$$

$$=\mathbf{e}_z\frac{\mu_0}{\pi}\frac{I}{kr}e^{ikr}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$



From (3), $\left\langle\frac{dP}{d\Omega}\right\rangle_t=\frac{Z_0}{2\mu_0^2}k^2r^2\sin^2\theta|\mathbf{A}|^2$

$$=\frac{Z_0I^2}{2\pi^2}\cos^4\left(\frac{\pi}{2}\cos\theta\right)/\sin^2\theta \quad \left[\begin{array}{l}\text{same as the full wave}\\ \text{solution in (9.57)}\end{array}\right]$$

Solution to problem 2:

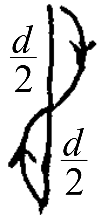
If the two half-wave antennas in problem 1 are excited 180° out of phase, we simply replace the "+" sign in (6) with a "−" sign.

9.4 Center-fed Linear Antenna (*continued*)

Thus,

$$\mathbf{A} = \mathbf{A}_1 - \mathbf{A}_2 = \mathbf{e}_z \frac{\mu_0}{2\pi} \frac{I}{kr} e^{ikr} [e^{-i\frac{\pi}{2}\cos\theta} - e^{i\frac{\pi}{2}\cos\theta}] \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

$$= -i\mathbf{e}_z \frac{\mu_0}{\pi} \frac{I}{kr} e^{ikr} \frac{\sin(\frac{\pi}{2}\cos\theta)\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$



antenna 1 $\frac{d}{2}$

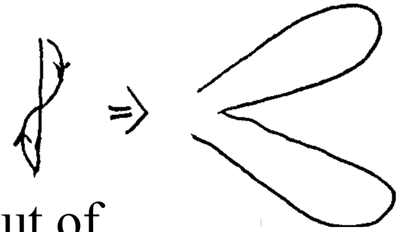
antenna 2 $\frac{d}{2}$

From (3), $\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{Z_0}{2\mu_0^2} k^2 r^2 \sin^2\theta |\mathbf{A}|^2$

$$= \frac{Z_0 I^2}{2\pi^2} \frac{\sin^2(\frac{\pi}{2}\cos\theta)\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} = \frac{Z_0 I^2}{8\pi^2} \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$

Solution to problem 3:

in phase ⇒ dipole radiation



out of phase ⇒ quadrupole radiation

Question: How does a phased array antenna work?

Homework of Chap. 9 Problems: 2, 3, 6

Problem 9.2

A radiating quadrupole consists of a square of side a with charges $\pm q$ at alternate corners. The square rotates with angular velocity w about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of radiation, and the total radiated power, all in the long-wavelength approximation. What is the frequency of the radiation?

Problem 9.3

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V = \cos \omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

Problem 9.6

(a) Starting from the general expression (9.2) for **A** and the corresponding expression for Φ, expand both $R = |\mathbf{x} - \mathbf{x}'|$ and $t' = t - R/c$ to first order in to obtain the electric dipole potentials for arbitrary time variation where $|\mathbf{x}|/r$ is the dipole moment evaluated at the retarded time measured from the origin.

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r^2}\hat{\mathbf{n}}\cdot\mathbf{p}_{ret} + \frac{1}{cr}\hat{\mathbf{n}}\cdot\frac{\partial \mathbf{p}_{ret}}{\partial t}\right]; \qquad \mathbf{A}(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\frac{\partial \mathbf{p}_{ret}}{\partial t}$$

where $\mathbf{p}_{ret} = \mathbf{p}(t' = t - r/c)$ is the dipole moment evaluated at the retarded time measured from the origin.

(b) Calculate thedipole electric and magnetic fields directly from these potentials and show that

$$\mathbf{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\left[-\frac{1}{cr^2}\hat{\mathbf{n}}\times\frac{\partial \mathbf{p}_{ret}}{\partial t} - \frac{1}{c^2 r}\hat{\mathbf{n}}\times\frac{\partial^2 \mathbf{p}_{ret}}{\partial t^2}\right]; \quad \mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\varepsilon}\left\{\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}_{ret}) - \mathbf{p}_{ret}}{r^3}\right] + \frac{1}{c^2 r}\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\frac{\partial^2 \mathbf{p}_{ret}}{\partial t^2}\right)\right\}$$

(c) Show explicitly how you can go back and forth between these results and the harmonic fields of (9.18) by the substitutions $-i\omega \leftrightarrow \partial/\partial t$ and $\mathbf{p}e^{ikr - i\omega t} \leftrightarrow \mathbf{p}_{ret}(t')$.

Homework of Chap. 9 Problems: 14, 16, 17

Problem 9.14

An antenna consists of a circular loop of wire of radius a located in the x - y plane with its center at the origin.
The current in the wire is

$$I = I_0 \cos \omega t = \mathrm{Re}\, I_0 e^{i\omega t}$$

(a) Find the expressions for **E**, **H** in the radiation zone without approximations as to the magnitude of ka.
Determine the power radiated per unit solid angle.

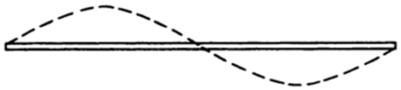
(b) What is the lowest non-vanishing multipole moment ($Q_{l,m}$ or $M_{l,m}$) ? Evaluate this moment in the limit $ka << 1$.

Problem 9.16

A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure.

(a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

(b) Determine the total power radiated and find a numerical value for the radiation resistance.



Problem 9.16

Problem 9.17

Treat the linear antenna of Problem 9.16 by the multipole expansion method.

(a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long-wavelength approximation.

(b) Compare the shape of the angular distribution of radiated power for the lowest non-vanishing multipole with the exact distribution of Problem 9.16.

(c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part a. Compare with Problem 9.16b. Is there a paradox here?

Homework of Chap. 9 Problems: 22, 23

Problem 9.22

A spherical hole of radius a in a conducting medium can serve as an electromagnetic resonant cavity.

(a) Assuming infinite conductivity, determine the transcendental equations for the characteristic frequencies ω_{lm} of the cavity for TE and TM modes.

(b) Calculate numerical values for the wavelength λ_{lm} in units of the radius a for the four lowest modes for TE and TM waves.

(c) Calculate explicitly the electric and magnetic fields inside the cavity for the lowest TE and lowest TM mode.

Problem 9.23

The spherical resonant cavity of Problem 9.22 has non-permeable walls of large, but finite, conductivity. In the approximation that the skin depth δ is small compared to the cavity radius a, show that the Q of the cavity, defined by equation (8.86), is given by

$$Q = \frac{a}{\delta} \quad \text{for all TE modes}$$

$$Q = \frac{a}{\delta}\left(1 - \frac{l(l+1)}{x_{lm}{}^2}\right) \quad \text{for TM modes}$$

where $x_{lm}{}^2 = \frac{a}{c}\omega_{lm}$ for TM modes.